

4. The Orbit

After release: the inclination, the survival and the recession
from the detachment conditions

Parts A--E · Appendices A--B · $\lambda = 22.5^\circ$

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The prose is written to be read by a general readership.

The environment from which the calculation starts

The paper begins at the end of the conjectured construction. In the LEF model, material from the Earth builds an attached body by feed from within, and the outer shell rises with it. The body sits as a ball squashed on the tip of the pear: its contact with the pear peels from about 6,000 km across to 3,000 km, and then it detaches, rounded and rigid. Paper 2, *LEF Mechanism in Principle*, examines the angular-momentum budget, corotation and the release conditions; Paper 3, *LEF Mechanism in Practice*, deals with the construction and the connection. This paper takes over at the moment the Moon is released. It examines the orbit produced by those initial conditions and the requirements for its evolution into the present orbit.

Because the system changes during the process — mass passes from one body to the other, the Earth's spin and shape change — every number below belongs to a defined moment. The table has three columns: the first names the domain, the second gives the state or value used in this paper, and the third says what kind of number it is and how it may be used. The third column lets the reader see at once which values are put in as assumptions of the scenario and which are results carried over from Papers 2 and 3.

Domain	State or value used here	Status and meaning
Time	Start of the ancient system; the orbital clock begins at detachment	The time of detachment is not redetermined in this paper
Duration of construction	A scenario of up to about 600 million years in the series, from the start of the bulge to release at 3.873 Ga (roughly 400 Myr for the bulge and the pear, about 100 Myr of flattening, about 100 Myr to release)	Scenario value; not measured from the age difference between maria and crust, and not the duration of recession after detachment
Initial mass	$M_0 = M_{\oplus} + m = 6.0455 \times 10^{24} \text{ kg}$	Before the division; the lunar mass must not be added a second time
Masses after the division	$M_{\oplus} = 5.972 \times 10^{24},$ $m = 7.35 \times 10^{22} \text{ kg}$	The rounded values shared by the series
Orbital gravity	$K = G(M_{\oplus} + m) = 4.0347667 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$	The relative separation motion of the two bodies
Rotation at the start of construction	5.5 hours in the reference scenario	This period must not be inserted automatically at the moment of detachment
Parent structure	The figure and moment-of-inertia model of Paper 2; the core centralises during the transfer, $D = 2$ at release	Core organisation changes the moment of inertia, the spin and the non-spherical field

Domain	State or value used here	Status and meaning
Dimensions of the ancient parent	Volumetric radius 6,517 km; initial equatorial radius about 6,721 km	The equatorial figure is recomputed with the spin in Paper 2
Comparative distance unit	$R_{\oplus, \text{now}} = 6,371 \text{ km}$	A reporting unit only, not the radius of the ancient Earth
Release latitude	$\lambda = 22.5^\circ$, the latitude of the centre of the contact area on which the Moon sat	The latitude of the centre of mass relative to the spin axis at the time of release: the contact centre, at the tip of the pear. The base of the pear stood near Kiritimati, $1.9^\circ\text{N } 157.5^\circ\text{W}$, roughly on the equator of that time (Paper 3); $25^\circ\text{N } 165^\circ\text{E}$ is the back of the pear
Release state	$r_0 = 17,496 \text{ km}$, $T_{\oplus} = 5.77 \text{ h}$, $\lambda = 22.5^\circ$	The Moon begins to separate as its centre of mass reaches the line where the speed carried by the rotation, $\omega r \cos \lambda$, equals the orbital speed (17,287 km; Paper 2, §B.1), and is released about 210 km beyond it, when a little more than half of it (about 58%) has crossed; the mechanism of release and the release-distance search are Paper 3 (§C.5–C.5a)
Density of the released body	$2,600 \text{ kg m}^{-3}$, radius about 1,890 km	Assumption of the survival test; not the density of the present Moon
Heat, volatiles and atmosphere	A hot body, cooling through the atmosphere, volatiles in the material and the environment	Series conditions; no temperature, vapour pressure or orbital drag measured for the period
Rheology	Envelope and connection of varying strength; k_2/Q depends on state and frequency	The viscosity of a sliding layer must not be identified with the tidal dissipation of a whole body

Domain	State or value used here	Status and meaning
Connection	A ball squashed on the tip of the pear; the contact peels from about 6,000 km across to 3,000 km, and separation occurs from this broad, hot, viscous contact	The body begins to separate when its centre of mass reaches the latitude-corrected corotation radius and is released about 210 km beyond it; the contact resists little, and the body is pulled outward with force; the separation itself is described in Paper 3
Target state	$a = 384,400$ km, $e = 0.0549$, $i_{\text{ecl}} = 5.145^\circ$, recession about 3.8 cm/yr	Modern comparison data [S1]

Some rows need a word of explanation. The initial mass, 6.0455×10^{24} kg, is today's Earth and today's Moon added together: before the division the lunar material is part of the parent, and afterwards the same mass is the Moon, so it is counted once. K is the strength of the gravity that governs the two bodies' motion about each other; it uses the sum of both masses, because each body pulls the other. The 5.5-hour day belongs to the start of construction; by the moment of release the Earth day is 5.77 hours, because the lengthening pear slowed the Earth and the iron gathering towards its centre ($D = 2$, the ratio of core density to mantle density) sped it up again (Paper 2). The present Earth radius, 6,371 km, is used only as a ruler for reporting distances, not as the size of the young Earth, which was hot and larger. The *release latitude* needs care, because three places are named in the series. The base of the pear — the place where the pear rose from the Earth's surface — stood near Kiritimati, $1.9^\circ\text{N } 157.5^\circ\text{W}$ (Paper 3), roughly on the equator of that time; whether that equator is today's is left to the study of the Earth–Moon axes, future work. The coordinate $25^\circ\text{N } 165^\circ\text{E}$, also from Paper 3, is read as the back of the pear. The latitude that matters for the orbit is neither of these: it is the latitude of the *centre* of the 3,000-km contact area on which the Moon sat at the moment of release, at the tip of the pear: 22.5° . The last row is the destination: the Moon's present mean distance a , its eccentricity e (how far the orbit departs from a circle; 0 is a perfect circle) and its inclination to the ecliptic, the plane of the Earth's orbit round the Sun.

Gravity does not “split” into two laws: during the transfer the mass distribution changes. Once the two masses are fixed, the motion of the centres in the monopole approximation — treating each body as if all its mass sat at its centre — is set by K , while reorganisation of the core acts mainly through the moment of inertia, spin and higher gravitational moments (the departures of the field from that of a point, caused by the bulge and the internal layering). A connection of significant mass requires removing that mass from the other reservoirs and recomputing the system; it is not a free addition.

Abstract

In the LEF model, the Moon was built from terrestrial material while attached to the parent body, and then released. This paper examines the orbital outcome of such a release. The *inclination* of an orbit is the angle between the plane in which the body goes round and a reference plane — here the Earth’s equator at that time. A body whose centre lies at latitude λ , and whose tangential velocity at the moment of release is in the direction of rotation only, receives an initial inclination $i_0 = |\lambda|$ relative to the equator of that time. Local release at an intermediate latitude therefore provides a geometric origin for the initial inclination. Two remarks fix what λ is. It is the latitude of the body’s *centre* at the moment of release, not of any point on its base: a body 3,000 km across has one velocity vector, and the extent of its base does not spread the inclination. And it is the latitude of the centre of the 3,000-km contact area on which the Moon sat, at the tip of the pear: 22.5° , the latitude used throughout this paper. 25° serves as a sensitivity case — a higher latitude gives a larger initial inclination, and more of it to remove on the way to the present 5.1° (under the tan-law of Part D, 22.5° returns today’s 5.145° from the adopted release, and 25° returns 5.79° ; an earlier version, with release about 100 km beyond the line, gave 5.079° from 22.5°).

Combined with Paper 2, the release state is fixed: the Earth day at release is 5.77 h (the lengthening pear and the centralising core act together) and the Moon’s centre lies at latitude 22.5° . Off the equator the rotation carries the body at $\omega r \cos \lambda$, so the body reaches orbital speed at $r_c \cos^{-2/3} \lambda \simeq 17,287$ km ($x = 1.0542$), the corotation line for its latitude. The Moon begins to separate as its centre of mass reaches this line and is released about 210 km beyond it, at $r_0 \simeq 17,496$ km ($x = r_0/r_c = 1.0669$), when a little more than half of it has crossed; the contact resists little, and the body is pulled outward with force. Released there, it enters a nearly circular orbit ($e_0 \simeq 0.04$) inclined by 22.5° to the equator of that time, with the release point as perigee; the tide pushes outward from the first moment, so the Moon recedes from the start. The Moon at release is round, with a rigid outer shell; it lies about 7,200 km outside the rigid Roche limit of 10,314 km. The release distance is the one at which two reduced tests — the present inclination and the present eccentricity — meet (Paper 3, §C.5a).

Tests of the Laplace-plane transition, tidal dissipation and thermal history are then presented. The solution of the reduced decay equation refers to $\tan i$, not to $\sin i$; the explicit cancellation of $Q_\oplus$ in that equation does not cancel the dependence of the full orbit on cooling and resonances. The established result is the geometric source of inclination and the mapping from release conditions to initial orbits. A joint reconstruction of the present distance, inclination, eccentricity and recession rate remains to be computed.

WHAT THIS PAPER PROPOSES

The tilt of the Moon's orbit is the latitude it left from. A body released while turning with the Earth at latitude λ enters an orbit inclined by exactly λ to the Earth's equator: $\mathbf{h} = \mathbf{r} \times \mathbf{v}$, three lines of algebra, no parameter to adjust. A disk cannot do this — a disk settles onto the equator — which is why the lunar inclination has been an open problem for thirty years and why every remedy for it adds something to the picture. Released from 22.5° , the Moon begins 22.5° tilted; carried outward by the tides, the reduced law $\tan i \propto a^{-1/2}$ brings that to 5.145° at the present distance, the observed value. From 25° , a sensitivity case, the same law gives 5.788° .

The same geometry says where the body must have been released, and the paper says so. At the equatorial corotation radius a body at latitude λ moves too slowly for a circle: its orbit is eccentric ($e_0 = \sin^2 \lambda$), released at apogee, and the Earth's tide at first draws it in. Off the equator the speed carried by the rotation is $\omega r \cos \lambda$, so the corotation line lies a little farther out, at $x = 1.0542$, 17,287 km. The Moon begins to separate as its centre of mass reaches this line and is released about 210 km beyond it, at $x = 1.0669$, $r_0 = 17,496$ km, when a little more than half of it has crossed: the contact resists little, and the body is pulled outward with force. Released there, the Moon enters a nearly circular orbit ($e_0 \simeq 0.037$) whose perigee is the release point; the tide pushes outward from the first moment, so it recedes from its first orbit, about 7,200 km outside the rigid Roche limit of 10,314 km. (An earlier version, with release about 100 km beyond the line, gave $r_0 = 17,387$ km and $e_0 \simeq 0.017$; one before it used release farther beyond the line, at $x \simeq 1.15$.) The Moon at release is round, its outer shell rigid (axis ratio at most 1.04, Paper 2, §B.7); its flattening and rounding took place while it was attached (Paper 7). Paper 3 describes the separation from the contact, and Paper 2 finds the same release state from the side of the angular-momentum budget. What the reduced law does not do is the whole history: the Laplace-plane transition, the Cassini state, the fossil figure and a dissipation that must have varied — one constant Q fits either the present rate or the age, not both. Each of these is stated in Part E with the test that will check it. The requirements of the model, and what would test it, are collected in Paper 11 (The Requirements).

The box makes two claims, and it helps to see them in plain terms. The first is about direction. Material that turns with the Earth at latitude 22.5° moves eastward along its circle of latitude, parallel to the equator; but the orbit it enters after release must pass through the Earth's centre, because that is where gravity pulls. A plane that passes through the Earth's centre and through a point 22.5° north of the equator, and runs eastward there, is a plane tilted 22.5° to the equator. Nothing else is needed: no mass, no distance, no spin rate enters the answer. That is the meaning of "three lines of algebra, no parameter": the result is written down in three lines (equations 1–3 of Part A), and there is no number in it that could be tuned to fit. In the disk picture of the impact model, by contrast, the Moon forms from debris that spreads into a flat ring over the equator, so it starts with almost no tilt, and the present tilt must be produced afterwards by an added process.

The second claim is about distance. On the equator, the corotation radius is the distance at which the speed given by the spin equals the speed of a circular orbit; for the 5.77-hour day it is 16,399 km. Off the equator a body is closer to the spin axis

than to the Earth's centre, by the factor $\cos \lambda$, so the spin carries it more slowly than an equatorial body at the same distance. It therefore has to be carried a little farther out before its speed matches the orbital speed: at 22.5° this is 17,287 km, 5.42% beyond the equatorial line ($x = 1.0542$). A body released exactly on this line would enter a circle, and the tide would be neutral: the Moon would go round exactly as fast as the part of the Earth's spin that acts along its orbit, so the tidal bulge it raises on the Earth would be carried neither ahead of it nor behind it, and would give no push. The Moon begins to separate as its centre reaches this line and is released about 210 km beyond it, at 17,496 km. There it moves a little faster than a circle needs, so the release point is the nearest point of a nearly circular orbit ($e_0 \simeq 0.037$); and the Earth turns faster than the Moon goes round, the bulge is carried ahead of the Moon, and its pull draws the Moon forward along the orbit, so the orbit grows from the first moment. The *Roche limit* is the distance inside which the Earth's tide would pull a body apart; for a body held together by a rigid shell it is 10,314 km, and the Moon is released about 7,200 km outside it. The reduced law $\tan i \propto a^{-1/2}$ then describes how the tilt shrinks as the orbit widens: by the present distance 22.5° has become 5.145° , the observed value. Part D shows that this law is a reduced one; the full path through the Laplace-plane transition and the Cassini state, and a dissipation that changed with time, are the requirements listed in Part E.

Which inclination results at the moment of release?

THE PROBLEM AND THE SOLUTION

The reader meets two different angles: the release latitude on the parent body and the orbital inclination relative to the ecliptic. We first derive the inclination relative to the ancient equator; we then examine the transition to the present observational frame.

This part answers one question: at the moment the Moon is released, how steeply is its new orbit tilted? It first places the question among other models (A.1), then derives the answer (A.2), and finally separates three angles that are easily confused when the Earth's axis and the Moon's orbit are considered together (A.3). The passage from the ancient equator to the present ecliptic is the subject of Part C.

First, what the word means. Every orbit lies in a plane: the body goes round in that plane, and the Earth's centre lies in it. The *orbital inclination* is the angle between that plane and a chosen reference plane. Two reference planes appear in this paper. One is the Earth's equator at the time of release, which is natural at the start, when the Moon has just left the turning Earth. The other is the *ecliptic*, the plane of the Earth's orbit round the Sun, against which the Moon's present inclination of 5.145° is measured. An equivalent description, which the derivation uses, is through the direction perpendicular to the orbital plane: the inclination is the angle between that perpendicular and the perpendicular to the reference plane — for the equator, the Earth's spin axis.

A.1 The context of other models

Formation from a disk close to the equatorial plane raises the question of the origin of the present inclination. In the giant-impact family, the debris thrown out by the collision settles into a disk over the Earth's equator, and a Moon that grows from that disk orbits almost in the equatorial plane. The present tilt then has to be supplied later. However, the literature includes late excitation mechanisms, such as encounters with planetesimals [S2] and the coupled evolution of a system with high axial tilt [S3]. Formation from a disk therefore does not rule out later excitation of inclination. The difference examined here is that LEF can begin with a substantial inclination already at release, without needing late excitation to produce the starting angle. In LEF the tilt is not an added event; it follows from where on the turning Earth the body was

attached.

A.2 The derivation

Before the algebra, the picture. Imagine a merry-go-round whose middle stands at the Earth's centre, and imagine that the only pull on anything near it is towards that central point. A stone is carried round on a ring set at a height above the middle, like a raised row of seats. While it is carried, the stone moves horizontally — along its ring, parallel to the floor. Now let it go. It leaves with that horizontal velocity, but the pull on it is towards the central point below and inward. Its free path must therefore lie in the one plane that contains both the central point and the stone's velocity: a plane that runs from the stone, along its direction of motion, and dips down through the centre. That plane is tilted to the floor by exactly the angle at which the stone is seen from the centre, above the floor. A stone on a higher ring is seen at a steeper angle and goes into a steeper orbit; a stone on the floor itself stays in the floor's plane. For the Earth, the floor is the equator, the ring is the circle of latitude, and the angle at which the body is seen from the centre is its latitude λ . The equations below say the same thing in three lines.

Denote the spin axis at the moment of release by the z axis, and choose the x axis so that the body lies in the x - z plane; the y axis then points east at the body. For an eastward corotation velocity, in an inertial frame (one that does not turn with the Earth) and with separation between the centres of mass:

$$\mathbf{r} = r(\cos \lambda, 0, \sin \lambda), \quad \mathbf{v} = \omega r(0, \cos \lambda, 0). \quad (1)$$

Here $\mathbf{r}$ is the position of the body's centre relative to the Earth's centre, at distance r and latitude λ ; $\mathbf{v}$ is its velocity, and ω the Earth's spin rate. The body's distance from the spin axis is $r \cos \lambda$, so the spin carries it eastward at $\omega r \cos \lambda$; this is the speed that appears in the environment table.

The specific angular momentum — the angular momentum per unit mass, $\mathbf{h} = \mathbf{r} \times \mathbf{v}$ — is a vector that stands perpendicular to both $\mathbf{r}$ and $\mathbf{v}$, and therefore perpendicular to the orbital plane. Its direction is the orbit's "axis", and under a central pull it does not change after release. The specific angular momentum is:

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = \omega r^2(-\sin \lambda \cos \lambda, 0, \cos^2 \lambda). \quad (2)$$

Its length is $|\mathbf{h}| = \omega r^2 \cos \lambda \sqrt{\sin^2 \lambda + \cos^2 \lambda} = \omega r^2 \cos \lambda$. The inclination to the equator is the angle between $\mathbf{h}$ and the spin axis z , and its cosine is the z part of $\mathbf{h}$ divided by its length. Hence, for prograde motion (going round in the same sense as the Earth's spin) not at the pole:

$$\boxed{\cos i_0 = \frac{h_z}{|\mathbf{h}|} = \cos \lambda, \quad i_0 = |\lambda|.} \quad (3)$$

In the division, the factor ωr^2 appears both above and below and cancels; only the latitude is left. The condition "not at the pole" is needed because at the pole the body lies on the spin axis, is carried at no speed, and has no orbit to speak of.

This is a kinematic identity: it follows from the geometry of motion alone, not from any property of the material or any force law beyond a central pull. The mass, the distance and the rotation rate do not appear in the final result. However, the choice of latitude is an initial condition of the model, and the relation establishes neither the proposed source location nor the occurrence of detachment. What the identity does is to make the latitude and the tilt the same quantity: given one, the other is fixed.

The identity assumes that the velocity at release is purely eastward, as corotation gives it. Two other velocity components are possible, and they act differently. An additional radial velocity (straight outward or inward, along $\mathbf{r}$) does not change $\mathbf{h}$, because the cross product of $\mathbf{r}$ with a vector along $\mathbf{r}$ is zero, and therefore does not by itself change the orbital plane; it changes the shape of the orbit, not its tilt. If there is also a northward velocity v_N , alongside an eastward velocity $v_E > 0$:

$$\cos i_0 = \frac{v_E \cos \lambda}{\sqrt{v_E^2 + v_N^2}}. \quad (4)$$

Since v_N enters only as its square, a northward or a southward component both make $\cos i_0$ smaller, that is, the orbit steeper than λ ; with $v_N = 0$ equation (4) returns equation (3).

Separation from a broad connection therefore fixes the velocity of the centre of mass after the internal forces end. In the model the Moon begins to separate as its centre of mass reaches the corotation line and is released about 210 km beyond it; the contact resists little, and the body is pulled outward with force. That pull acts along the radius, and, as shown above, a radial component changes the shape of the orbit, not its tilt. The body leaves with the eastward velocity carried by the rotation, and the inclination at release is the latitude of its centre, 22.5°.

A.3 Three angles not to be confused

So far only the orbit has been considered. But the Earth also carries angular momentum, in its spin, and the Moon's orbit and the Earth's spin together make up the angular momentum of the whole system. Once the body is free and no outside torque acts, it is this total — not the Earth's axis and not the orbit's axis — that keeps a fixed direction in space. Three directions are therefore in play: the Earth's spin axis, the perpendicular to the Moon's orbit, and the total. This section gives the angles between them, and says what they are not.

At the release state (Paper 2: Earth spin 2.876×10^{34} , lunar orbit $5.90 \times 10^{33} \text{ kg m}^2 \text{ s}^{-1}$), 22.5° is obtained between the orbit and the Earth's axis, about 3.78° between the Earth's axis and the total angular momentum, and about 18.72° between the orbit and the total angular momentum. These are angles in the same vector triangle. The orbit carries about a fifth of the Earth's spin angular momentum, so the total lies much closer to the Earth's axis than to the orbit: the 22.5° between the two is shared as 3.78° on the Earth's side and 18.72° on the orbit's side ($3.78 + 18.72 = 22.5$). Neither of the latter two is by itself the inclination relative to the ecliptic, or an explanation of the 23.44° of the Earth's axis today. How the Earth's axis and the Moon's orbit evolved together after release, towards the present 23.44°, is not solved in this series; it is future work.

The division is described, neglecting a small lunar spin and at a stage without a significant external torque, by:

$$\mathbf{J} = \mathbf{S}_{\oplus} + \mu_r \mathbf{h}, \quad \mu_r = \frac{M_{\oplus} m}{M_{\oplus} + m}. \quad (5)$$

In words: the total angular momentum $\mathbf{J}$ is the Earth's spin angular momentum $\mathbf{S}_{\oplus}$ plus the orbital angular momentum of the pair. The orbital part is the specific angular momentum $\mathbf{h}$ of equation (2) multiplied by the *reduced mass* μ_r , the standard mass that describes two bodies going round their common centre; because the Moon is so much lighter than the Earth, μ_r is very close to the Moon's own mass.

The calculation establishes consistency between states, rather than a sudden “kick” to the axis at detachment: the exchanges of angular momentum can accumulate during construction. In other words, the Earth's axis is not knocked over when the Moon leaves; the angles of the triangle describe how the angular momentum is already divided between spin and orbit at that moment. In a full treatment lunar spin, ejected material and external torques must also be included.

Figure 4 draws the triangle. Read it from the common origin: the navy arrow is the Earth's spin, the red arrow is the orbit, tilted 22.5° from it, and the black arrow is their sum. The dashed lines complete the parallelogram of the vector addition. Because the orbit arrow is the shorter one, the sum leans only a little away from the Earth's axis.

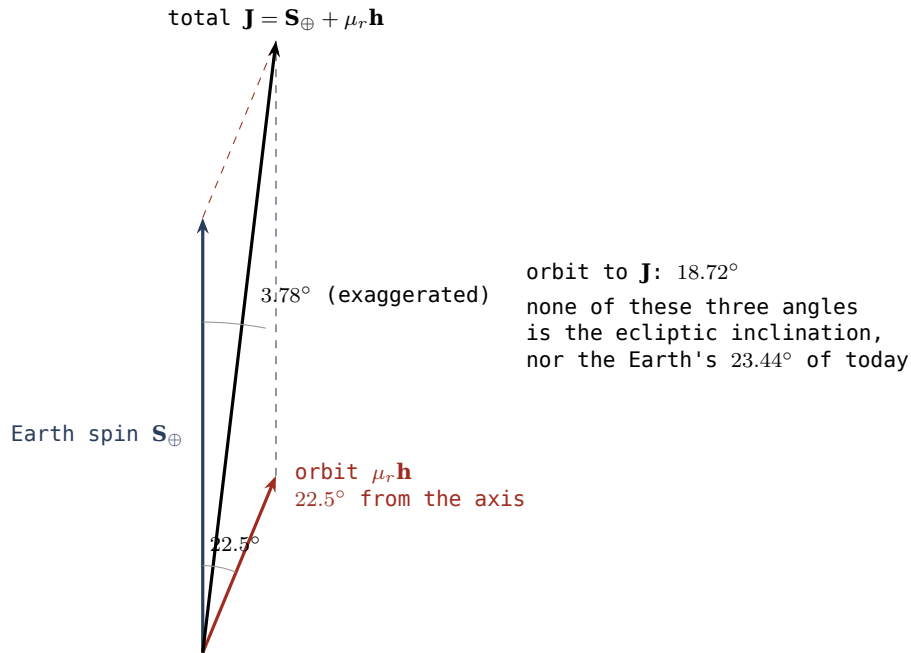


Figure 4. The three angles of Part A.3 in one vector triangle (not to scale; the orbit vector and the 3.78° are exaggerated): the orbit stands 22.5° from the Earth's axis, the axis 3.78° from the total angular momentum, and the orbit 18.72° from the total. The ecliptic inclination is measured against a fourth direction, and requires Part C.

The fourth direction named in the caption is the perpendicular to the ecliptic, set by the Earth's motion round the Sun, not by anything in the Earth–Moon pair. What Part A has established is the starting tilt: 22.5° to the equator of the time of release, fixed by the latitude of the body's centre. Part B finds the size and shape of the orbit the

body enters at its release point, and Part C follows the orbital plane outward to the frame in which today's 5.145° is measured.

PART B

Does the body enter an orbit that survives and recedes?

THE PROBLEM AND THE SOLUTION

The corotation velocity is not necessarily the circular velocity appropriate to the release latitude. We compute the resulting ellipse, and connect it to the tidal and survival tests already developed in Paper 2.

Part A showed that a body released while turning with the Earth at latitude λ enters an orbit tilted by λ to the Earth's equator. This Part asks the next question: what is the *shape and size* of that orbit, does the Earth's tide then push it outward or draw it inward, and is the body far enough from the Earth to stay whole? The point matters because an inclination is of no use if the orbit that carries it spirals into the Earth or tears the body apart.

The opening sentence of the box needs one word of explanation. A body turning with the Earth at latitude λ does not circle the Earth's axis at its full distance r from the centre, but at the smaller distance $r \cos \lambda$ from the axis. It therefore moves more slowly than a body at the same distance over the equator, and at the equatorial corotation radius it is slightly too slow for a circular orbit. The orbit it enters is then an ellipse, and the release point has to be placed so that this is taken into account; this section finds where the orbit is a circle and what orbit the Moon enters at its actual release point, a little beyond it.

B.1 From release coordinates to an ellipse

An orbit around the Earth is in general an ellipse. Three numbers describe its size and shape. The *semi-major axis* a is half the longest diameter of the ellipse, the average of the nearest and the farthest distance; it sets the orbital period. The *eccentricity* e measures how stretched the ellipse is: $e = 0$ is a circle, and values approaching 1 describe a long, narrow ellipse. The nearest point of the orbit to the Earth is the *perigee*, at distance $r_p = a(1 - e)$; the farthest point is the *apogee*, at $a(1 + e)$.

Denote:

$$B = \frac{\omega^2 r_0^3 \cos^2 \lambda}{K}, \quad \mathcal{E} = \frac{K}{r_0} \left(\frac{B}{2} - 1 \right). \quad (6)$$

Here ω is the angular velocity of the Earth's spin at release, r_0 the distance of the Moon's centre from the Earth's centre, λ its latitude and $K = G(M_\oplus + m)$ the gravitational parameter of the pair (G the gravitational constant, $M_\oplus$ and m the masses of the Earth and the Moon). The number B has a simple meaning: the body moves at the speed $v = \omega r_0 \cos \lambda$, and $B = v^2/(K/r_0)$ is the square of that speed divided by the square of the circular-orbit speed at the same distance. $B = 1$ means the body has exactly the speed of a circular orbit; $B < 1$ means it is too slow; $B > 1$, too fast; and $B = 2$ would be the escape speed. $\mathcal{E}$ is the orbital energy per unit mass; it is negative, so the body is bound to the Earth, as long as $B < 2$.

For release without radial velocity (the body moving neither toward nor away from the Earth at that instant), when $0 < B < 2$, in the two-point-mass approximation (the Earth and the Moon treated as point masses):

$$\boxed{a_0 = \frac{r_0}{2 - B}, \quad e_0 = |1 - B|, \quad r_p = a_0(1 - e_0).} \quad (7)$$

In words: the release distance and the single number B fix the whole initial orbit. The further B lies from 1, the more eccentric the ellipse.

If $B < 1$, the release is at apogee; if $B > 1$, at perigee; $B = 1$ gives a circle. The reason is direct. A body that is too slow for a circular orbit at its distance begins to fall inward, so the point where it was released is the farthest point of its orbit; a body that is too fast swings outward, so the release point is the nearest. At the radius $r_c = (K/\omega^2)^{1/3}$ — the equatorial corotation radius, where a body over the equator turning with the Earth has exactly the circular speed — a body at latitude λ has $B = \cos^2 \lambda$, and therefore $e_0 = \sin^2 \lambda$, not zero. The $\cos^2 \lambda$ is the slowing explained above: the body circles the axis at $r \cos \lambda$, not at r . The orbit is circular, $B = 1$, at

$$\boxed{r_0 = r_c \cos^{-2/3} \lambda,} \quad (7a)$$

where the speed carried by the rotation equals the orbital speed. Moving the release point outward makes up the missing speed, because the speed carried by the rotation grows with distance while the circular speed falls; equation (7a) gives the distance at which the two meet for a body at latitude λ . For a 5.77-hour day, $r_c = 16,399$ km (about 2.6 Earth radii), and at $\lambda = 22.5^\circ$ the circular radius — the corotation line for this latitude — is 17,287 km, about 900 km farther out, a factor $x = 1.0542$. The Moon begins to separate as its centre of mass reaches this line and is released about 210 km beyond it, at $r_0 = 17,496$ km ($x = 1.0669$), when a little more than half of it (about 58%) has crossed. There $B = 1.0367$, slightly above 1: the body is a little faster than a circle needs, the release point is the perigee, and equation (7) gives a nearly circular orbit with $e_0 = 0.037$, $a_0 = 18,163$ km and an apogee of 18,830 km. (An earlier version, with release about 100 km beyond the line, gave $r_0 = 17,387$ km, $e_0 = 0.017$, $a_0 = 17,696$ km and an apogee of 18,005 km.) The release distance is the one at which the reduced inclination and eccentricity tests of Paper 3, §C.5a, meet.

B.2 Combined results from Paper 2

The table below applies equations (6)–(7) to five release points and adds two tests: the direction in which the Earth's tide first moves the orbit, and the distance from the Roche limit. Here $x = r_0/r_c$, the release distance in units of the equatorial corotation

radius. All rows use the release day of Paper 2, 5.77 h ($D = 2$), and $\lambda = 22.5^\circ$. ($D = 2$ is the density ratio of core to mantle in the Paper 2 structure that gives this day.) The Roche column uses the rigid limit, since the body is rigid and round at release (Paper 2, §B.7).

D	x	Earth day at re- lease, h	r_0 , km	a_0 , km	e_0	r_p , km	Initial terrestrial tidal con- tribution	Rigid Roche limit, 10,314 km
2	1.00000	5.77	16,399	14,304	0.1464	12,209	contraction	passes
2	1.05420	5.77	17,287	17,287	0	17,287	neutral	passes
2	1.06695	5.77	17,496	18,163	0.0367	17,496	expansion	passes
2	1.08000	5.77	17,710	19,151	0.0752	17,710	expansion	passes
2	1.10000	5.77	18,038	20,880	0.1361	18,038	expansion	passes

How to read the table: the rows move the release point outward from top to bottom, and the tint shows the direction of the tide — blue for an orbit the tide draws inward, cream for the corotation line, where the tide is neutral, and red, deepening with distance, for orbits the tide pushes outward. In the first row ($x = 1$, on the equatorial line) the body is too slow, $e_0 = 0.1464 = \sin^2 22.5^\circ$, and the release point is the apogee: the orbit dips to a perigee of 12,209 km, well inside the release distance. In the second row, on the corotation line for 22.5° , the tide is neutral and the eccentricity is zero and all three distances coincide at 17,287 km; the circular orbit is a single, well-defined point. The row with bold figures is the adopted release, about 210 km beyond the line, where the tide expands the orbit from the first moment: the body is slightly too fast, the release point is the perigee ($r_p = r_0 = 17,496$ km), and the orbit is nearly circular, $e_0 = 0.0367$, with $a_0 = 18,163$ km. In the last two rows the body is faster still and the orbit swings out to a larger a_0 ; the eccentricity grows steadily on the far side of the circular point.

The release day of 5.77 hours follows from the initial 5.5-hour day: the lengthening pear slows the Earth and the centralising core speeds it up (Paper 2, §A.4).

The direction of the tide. The Moon raises two tidal bulges on the Earth, one facing it and one on the far side. If the Earth turned exactly as fast as the Moon goes round, the bulges would sit directly under the Moon, pull straight along the Earth-Moon line, and neither speed up nor slow down the orbit. If the Earth turns faster than the Moon goes round, friction inside the Earth carries the bulge slightly ahead of the Moon; the near bulge then pulls the Moon forward along its path, adds energy to the orbit, and the orbit grows — the Moon recedes, as it does today. If the Moon goes round faster than the Earth turns, the bulge lags behind it, pulls it back, and the orbit shrinks, as happens today to Mars’s inner moon Phobos. The test below decides which of these three cases applies at each release point.

The sign of tidal orbital evolution in the table was computed in a constant-time-lag model, by averaging the power over the ellipse; this is a defined test, not a rheology-independent rule. “Constant time lag” means that the bulge is assumed to respond to the tidal pull after a fixed short delay; “rheology” is the way the Earth’s material deforms and dissipates energy, which a full treatment would have to specify. Appendix

A gives the formula that reproduces it, in the framework of vector equilibrium tides [S4]. The formula, equation (14), compares two numbers: on the left, the rate at which the Earth turns about the axis of the orbit, $\omega \cos i$, divided by the mean orbital rate n ; on the right, a factor that depends only on the eccentricity and equals 1 for a circle. In the first row the comparison is $0.7526 < 1.1962$, so the terrestrial contribution to the evolution of a is negative: the Moon outruns the turning Earth on average, and the orbit contracts. On the corotation line, $x = 1.0542$, the two sides are equal: the orbit is synchronous and the tide is neutral. A body exactly on this line circles the Earth in the time the Earth takes to turn under it, so its bulge stands directly beneath it and exerts no torque along the orbit. The Moon is released about 210 km beyond the line, at $x = 1.0669$, and there the comparison is already $>$: $1.0769 > 1.0128$ (an earlier version, with release about 100 km beyond the line, gave $1.0357 > 1.0029$), so the terrestrial contribution is positive from the first moment (row $x = 1.08$: $1.1660 > 1.0532$). The Earth turns faster than the Moon goes round, the bulge leads, and the Moon is pushed outward. Once the Moon begins to recede, its orbital rate falls further below the Earth's spin, so the push continues; the neutral line is unstable in the outward direction, and the release state leads into recession. Tides in the Moon may change the overall balance; tides raised by the Earth inside the Moon are not included in this sign test (Appendix A).

The Roche limit. The Roche limit is the distance inside which the tidal pull of the Earth — the difference between its gravity on the near and the far side of the Moon — exceeds the Moon's own gravity and can pull it apart. The Roche test is made with the rigid limit,

$$r_{\text{Roche,rigid}} = 1.26 \left(\frac{3M_{\oplus}}{4\pi\rho_m} \right)^{1/3} \simeq 10,314 \text{ km}, \quad (8)$$

because the body is rigid and round at release (Paper 2, §B.7). Here ρ_m is the Moon's density; the denser the body, the closer it can approach without being torn. Every row lies outside it; the adopted release lies about 7,200 km beyond it (earlier versions, with release about 100 km beyond the line and on the line, gave 7,100 km and 7,000 km). Even the perigee of the first row, 12,209 km, stays outside the limit.

A specialist will ask why the rigid limit, and not the fluid one, is the right measure: for a body that deforms freely the limit lies farther out (about 19,971 km for $2,600 \text{ kg m}^{-3}$, Paper 2, §A.7.1), beyond the release radius. The answer lies in the state of the body. The Moon at release is round and its outer shell is rigid: its axis ratio is at most 1.04 (Paper 2, §B.7). The flattening and the later rounding of the lunar body took place earlier, while it was still attached (Paper 7, *The Poles*). A body with a rigid shell does not stretch along the tidal axis as a fluid drop would, and the rigid limit is the one that applies to it.

Figure 1 shows the same results as a continuous curve rather than four rows, so that the reader can see where the corotation line and the adopted release lie among all possible release points.

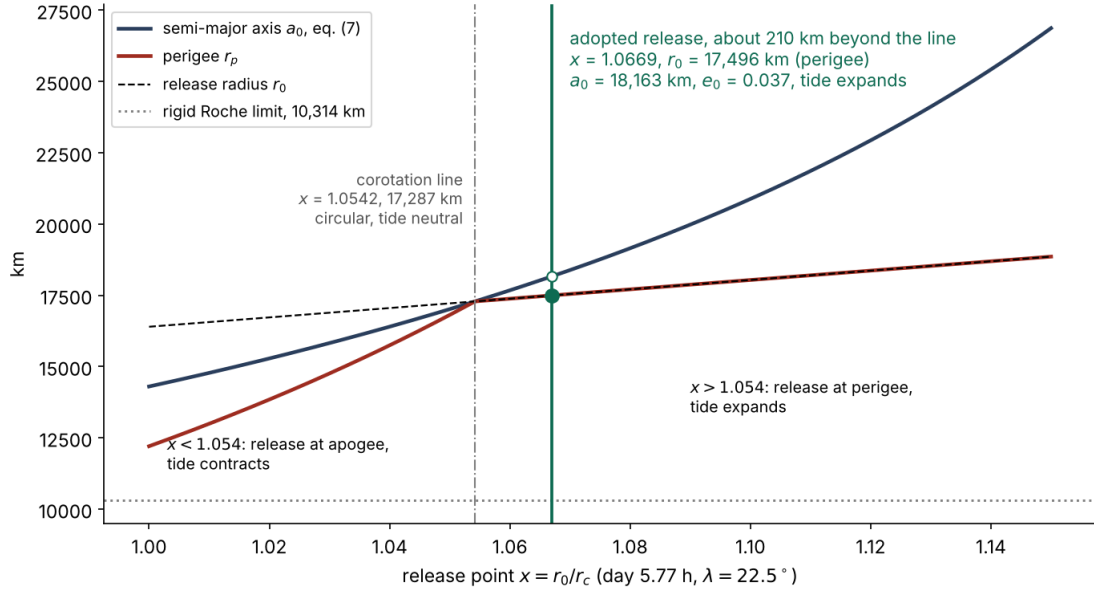


Figure 1. The initial orbit as a function of the release point at a 5.77-hour day and latitude 22.5° . Below $x = 1.054$ the body is released at apogee and the tide contracts the orbit; at $x = 1.054$, the corotation line, the orbit is circular and the tide is neutral; beyond it the body is released at perigee and recedes. The adopted release, about 210 km beyond the line at $x = 1.067$, is marked: perigee 17,496 km, $a_0 = 18,163$ km, $e_0 = 0.037$. Every point lies outside the rigid Roche limit.

B.3 The release state

The Moon begins to separate as its centre of mass reaches the latitude-corrected corotation radius, 17,287 km, and is released about 210 km beyond it, at 17,496 km, when a little more than half of it has crossed. The contact resists little, and the body is pulled outward with force. Released there, it moves slightly faster than a circular orbit needs: the release point is the perigee of a nearly circular orbit ($e_0 \simeq 0.037$, apogee 18,830 km), its perigee lies beyond the synchronous line, and the Moon recedes from the first orbit. If the outward pull also gave the body a small radial velocity, the eccentricity would be somewhat larger, but the tilt would not change (§A.2). The separation itself is described in Paper 3.

How the release distance enters. The distance beyond the line sets the shape of the first orbit and, through its size, the final angle. Moving the release outward raises the speed ratio B , and with it e_0 and a_0 ; the final angle of the reduced law grows slowly with a_0 . Paper 3, §C.5a, gives the full search; four of its rows at 22.5° show the trend:

Offset beyond the line (km)	e_0	a_0 (km)	Final angle
0	0	17,287	5.020°
100	0.017	17,696	5.079°
≈ 210 (adopted)	0.037	18,163	5.145° (observed)
500	0.089	19,531	5.334°

Table B.3. The initial orbit and the final angle of the reduced law as the release moves beyond the corotation line at 22.5° (Paper 3, §C.5a). The navy deepens with the final angle; green marks the adopted release, which returns the observed 5.145° .

Release exactly on the line would give a circle, which the tide carries outward as a circle; the present eccentricity therefore requires a release a little beyond the line. In Paper 3 the same offset, about 210 km, also returns today's eccentricity of 0.0549 for a hot young Moon; the two reduced tests meet there, and the full integration of Further work, item 1, carries the release state through the whole history. The three conditions asked for at the head of this Part are therefore met together: the orbit is nearly circular, the tide moves it outward, and the body lies well clear of the limit at which it would be torn apart.

RESULT

What is obtained is the **orbital starting point**: $a_0 \simeq 18,200$ km (18,163 km), $e_0 \simeq 0.037$ with the perigee at the release point, 17,496 km, and $i_0 = 22.5^\circ$ to the equator of that time. (Earlier versions, with release about 100 km beyond the line and on the line, gave $a_0 = 17,696$ km, $e_0 \simeq 0.017$, and $a_0 = 17,287$ km, $e_0 \simeq 0$.)

These three numbers are the input to everything that follows: Part C follows how the plane of this orbit turns as it moves outward, and Part D how much of the 22.5° survives to the present distance.

How does the orbital plane change on the way out?

THE PROBLEM AND THE SOLUTION

The initial inclination is measured relative to the ancient equator; today's inclination is measured relative to the ecliptic. We identify the forces that rotate the orbital plane, keeping the two reference planes distinct.

Part B ended with an orbit tilted 22.5° to the Earth's equator of that time. The observed 5.145° is measured against a different plane, the *ecliptic* — the plane of the Earth's orbit around the Sun. This Part explains why the reference plane changes as the Moon moves out, and what must be known to follow that change. It matters because comparing an angle measured from one plane with an angle measured from another would be comparing two different quantities.

C.1 Equator, ecliptic and the Laplace surface

An inclined orbit does not keep a fixed orientation in space. Outside forces twist it, so that the orbit's axis slowly circles around some reference direction, like the axis of a tilted spinning top circling the vertical; this slow turning is called *precession*, and the tilt is measured from the plane around which it occurs. Near the Earth, terrestrial oblateness strongly affects the rotation of the orbital plane: the Earth's equatorial bulge pulls the orbit toward the equator, and the orbit precesses about the equatorial plane. Farther out, the solar perturbation grows in importance, and the orbit precesses about the ecliptic, as the Moon's orbit does today. The Laplace surface describes equilibrium orientations of the orbital plane under both influences: at each distance it is the plane about which the orbit precesses. Close to the Earth it coincides with the Earth's equator; far out, with the ecliptic; in between it bends from one to the other. As the Moon recedes, the plane against which its tilt is effectively measured therefore turns from the equator to the ecliptic. The accepted transition scale, in the circular and quadrupole approximation, is [S5]:

$$r_L = \left(\frac{2J_2 R_\oplus^2 a_\odot^3 M_\oplus}{M_\odot} \right)^{1/5}. \quad (9)$$

Here J_2 is the number that measures the Earth's equatorial bulge in its gravity field, $R_\oplus$ the Earth's radius, $a_\odot$ the Earth-Sun distance and $M_\odot$ the Sun's mass. The equation

gives the distance at which the twisting effect of the Earth’s bulge and that of the Sun are about equal: inside r_L the bulge dominates, outside it the Sun. A larger bulge pushes the transition outward, though only weakly, through the fifth root.

This expression shows what must be known: the gravitational moment $J_2 R_\oplus^2$, not only the rotation period. The core, the shape and the cooling affect the relation between the two. A faster-spinning Earth bulges more, but how much more depends on how its mass is distributed inside.

An order-of-magnitude check only, with the present Earth radius and $J_2 = J_{2,\text{now}}(\omega/\omega_{\text{now}})^2$ — that is, assuming the bulge grows with the square of the spin rate, as rotational flattening does — gives about 17.07 Earth radii for the release day of 5.77 hours. The release radius of 17,496 km is about 16.1% of this scale (earlier versions, with release about 100 km beyond the line and on the line, gave 16.0% and 15.9%), roughly one-sixth of the way out to the transition (the $x = 1.08$ release of Part B, 17,710 km, would be about 16.3%). The Moon therefore begins its outward journey well inside the region where the Earth’s bulge controls the plane, and its 22.5° is measured, at first, against the equator. The check supports a start in the region where oblateness matters; it does not replace a calculation of J_2 from the ancient structure, and does not determine how the transition radius evolved with time.

C.2 When can the adiabatic approximation be used?

As the Moon recedes, the Laplace surface at its position tilts gradually from the equator toward the ecliptic. The question is whether the orbit follows that turning plane smoothly, keeping its tilt relative to it. An “adiabatic” change is one slow enough for the orbit to complete many precession cycles while the guiding plane turns only a little; in that case the orbit tends to follow the plane, much as a top keeps circling a support that is tilted slowly.

If the equilibrium surface changes slowly relative to the precession cycles, a conserved adiabatic action can be sought. This does not imply exact preservation of the inclination angle in every eccentric, finitely inclined orbit. The classical Laplace surface is defined for circular orbits; the instability threshold of about 68.875° also belongs to that defined framework [S5]. (That threshold is the tilt of the planet’s spin axis above which, in the transition zone, orbits lying on the classical Laplace surface become unstable.)

The required test combines the tilt of the Earth’s axis at that time, the line of nodes, the eccentricity, the recession rate and the evolution of the field. The line of nodes is the line along which the orbital plane crosses the reference plane; its orientation relative to the Sun decides in which direction the plane is twisted. The Earth’s tilt today is not a proven input for the past. How the Earth’s axis and the Moon’s orbit evolved together after release — the Earth–Moon axis system — is not solved in this series; it is future work. A transition at 14.8 Earth radii, or an “exact transfer” of the equatorial angle to the ecliptic angle, must therefore not be fixed from the reduced calculation.

C.3 The fossil figure and the Cassini state

Two further features of the present Moon record its passage outward. The *fossil figure* is the Moon's present shape, which is more flattened than its present spin and distance would produce; it is read as a shape frozen in when the Moon was closer and turned faster. A *Cassini state* is a configuration in which the Moon's spin axis, the axis of its orbit and the axis of the plane about which the orbit precesses stay in one plane while precessing together; as the Moon receded it passed from one such state to another.

The fossil figure of the Moon and the Cassini-state transition are additional constraints, not evolutionary stages whose orbital locations are fixed by the release latitude. A thermal-orbital model of Downey, Nimmo & Matsuyama found figure freezing before about 16 Earth radii under the conditions tested [S6]. Hence 21.6 Earth radii is not a unique freezing radius derived from observation, and the order "Laplace, figure freeze, Cassini" is not a universal result.

A comparison of two isolated precession rates also does not prove a new resonance at 17–19 Earth radii. Such a transition requires the frequencies of the coupled system, including the Sun and the response of both angular-momentum vectors. The paper does not claim that a channel for generating axial tilt is open only for the LEF path.

What we now have: the Moon starts with 22.5° measured from the Earth's equator, in the region where the equator governs the plane; the transfer of that tilt to the ecliptic, and the stages it passes on the way, are fixed by a full integration that includes the Earth's bulge, the Sun and the axes of both bodies. Part D turns to the amount of tilt that the tides remove on the way.

How much of the inclination survives, and in what time?

THE PROBLEM AND THE SOLUTION

A match to one angle is not enough if the calculation uses the wrong initial orbit or reaches the present distance at the wrong time. We solve the reduced decay model exactly and compare it with the thermal and orbital constraints.

The tides that push the Moon outward also act on the tilt of its orbit, reducing it as the Moon recedes. This Part takes the simplest law that describes this reduction, solves it exactly from the release state of Part B, and then checks it against the second requirement every orbital history must meet: arriving at the present distance in the time available.

D.1 What the decay equation actually says

Consider the illustrative law, with i measured from the equator of the parent (the transition to the ecliptic is Part C):

$$\frac{di}{da} = -\frac{1+\epsilon}{2a} \sin i \cos i. \quad (10)$$

In words: each small step outward, da , reduces the inclination by a small amount. The minus sign says that the tilt falls as the orbit grows; the factor $1/a$ says that equal *fractional* steps in distance count equally; and the factor $\sin i \cos i$ says that the reduction vanishes for an orbit in the equator ($i = 0$) and is largest at intermediate tilts.

Here ϵ is an additional-damping coefficient in a reduced model, not a solution for the lunar rheology. It stands for any extra loss of tilt, for example from dissipation inside the Moon; $\epsilon = 0$ means that only the basic tidal term acts. For a constant value the exact integral is:

$$\tan i(a) = \tan i_0 \left(\frac{a}{a_0} \right)^{-(1+\epsilon)/2}. \quad (11)$$

This is because $d \ln(\tan i)/di = 1/(\sin i \cos i)$. This is the *tan-law*: the tangent of the inclination falls as the inverse square root of the distance (for $\epsilon = 0$). If the distance

grows by a factor of four, the tangent of the tilt is halved. Replacing tangent by sine gives a different equation; it is a possible approximation at small angles only, where the two are nearly equal, and it is not accurate at a starting angle of 22.5° .

To illustrate, insert $i_0 = 22.5^\circ$, $\epsilon = 0$ and $a_f = 384,400$ km (the present mean Earth-Moon distance). From the starting semi-major axis, about 18,200 km, to 384,400 km the distance grows by a factor of about 21, so $\tan i$ falls by a factor of about 4.6:

Insertion condition	a_0 , km	Final angle in equation (11)	What may be inferred
Release state ($\lambda = 22.5^\circ$, about 210 km beyond the line, $e_0 \simeq 0.037$)	18,163	5.145°	Equal to the observed 5.145°; the orbit begins in recession (Part B)
Release 8% beyond the line	19,151	5.282°	Weak sensitivity to the release point
Same release at $\lambda = 25^\circ$	18,163	5.788°	Sensitivity to the latitude

The table changes one input at a time from the release state, and the deepening tint marks the growing departure of the final angle from the first row. The first row is the release state of Part B. The second uses the $x = 1.08$ release of the Part B table, 8% beyond the equatorial line, whose semi-major axis is 19,151 km: a semi-major axis about 5% larger than the adopted one changes the final angle by only about one-seventh of a degree. The third keeps the release distance and raises the starting latitude to 25° , as a sensitivity case; the final angle rises by about two-thirds of a degree. (Earlier versions, with release about 100 km beyond the line at $a_0 = 17,696$ km and on the line at $a_0 = 17,287$ km, gave 5.079° and 5.713° , and 5.020° and 5.648° , for the first and third rows.) The final angle is thus governed mainly by the release latitude, as equation (11) shows: it enters through $\tan i_0$, while the starting distance enters only through a square root.

All rows explore the same reduced equation. They do not integrate evolution through the Laplace-plane and Cassini-state transitions, do not include the evolution of the eccentricity and do not establish a prediction of the ecliptic inclination. In particular, the agreement of the final angle with 5.145° is a result of the reduced law for an orbit that begins in recession; it is not yet an integration through the transitions. The latitude at which this law returns 5.145° from the adopted release, $a_0 \simeq 18,200$ km, is the series latitude itself, 22.5° (earlier versions gave 22.8° with release about 100 km beyond the line, 23.0° with release on the line, and before that a range of 23.3° – 24.6°). Latitude and release distance trade against each other — half a degree of latitude moves the matching offset by about 200 km (Paper 3, §C.5a) — so the agreement is read together with the eccentricity test of Paper 3, not from the angle alone.

Figure 2 draws the tan-law as a curve from the release state to the present distance.

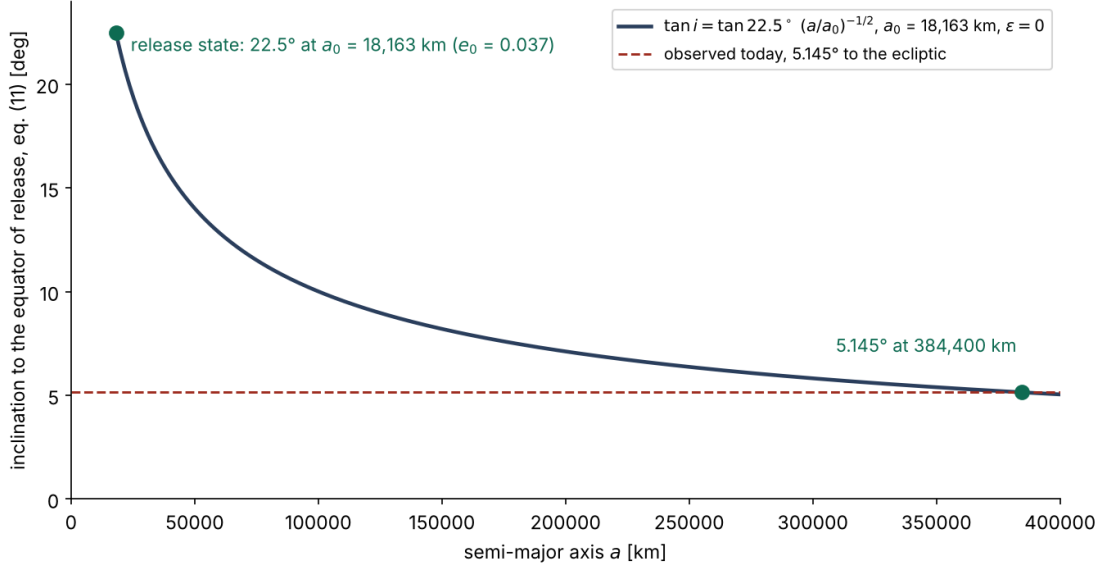


Figure 2. The reduced decay law of Part D from the release state ($a_0 = 18,163$ km, $e_0 = 0.037$, $i_0 = 22.5^\circ$): 5.145° at the present distance, the observed value. A single equation, no Laplace or Cassini transition, no eccentricity evolution.

The curve falls steeply near the Earth and flattens farther out: most of the tilt is lost in the first part of the journey, where each kilometre is a large fraction of the distance, and the last stretch to the present distance changes the angle little.

D.2 Why cancelling Q does not solve the thermal history

The strength of the tides is usually described by two numbers. The *Love number* k_2 measures how far a body deforms under a tidal pull: 0 for a perfectly rigid body, larger for a softer one. The *tidal quality factor* Q measures how much of the tidal energy is lost to friction in each cycle: a low Q means strong friction, a bulge carried far from the line to the Moon, and fast tidal evolution; a high Q means weak friction and slow evolution. $Q_\oplus$ is the value for the Earth.

When the same rate function multiplies both $\dot{a}$ and $\dot{i}$, it cancels in the division of the equations. This explains the explicit independence from $Q_\oplus$ within the illustrative law: equation (10) relates the tilt to the distance, not to time, so it gives the same final angle whatever the speed of the recession. But if ϵ depends on the melting state at the time of reaching a given distance, a change in the recession rate changes the melting state reached at that distance and hence the damping coefficient. An everyday picture: the route of a car between two cities does not depend on its speed, unless the roads change with time; then the speed decides which roads are open when the car reaches them. In addition, dissipation in the Moon need not be a constant multiple of the terrestrial tide.

Chen & Nimmo found in the models they tested a strong limit on the survival of the inclination, and a requirement of $Q_\oplus > 300$ in suitable scenarios [S7]. This is a model constraint, not a universal threshold for every material state. Solidification also does not switch off tides: a solid body can dissipate energy. Models including dissipation in the solid and in a magma ocean sharpen the conditions under which an ancient inclination survives [S6]. The thermal history of both bodies is therefore part of the inclination problem, and the reduced law, by cancelling Q , does not settle it.

D.3 One clock test suffices to expose the limit

The tan-law says nothing about time. The “clock test” supplies it: it asks whether a simple tidal law, run outward from the release state, brings the Moon to the present distance in the age of the system *and* with the present recession rate. Both must hold together.

We use, for illustration only, a circular-equatorial law in the outward direction, with constant parameters:

$$\dot{a} = Aa^{-11/2}, \quad A = 3 \frac{k_2}{Q_\oplus} \frac{m}{M_\oplus} \sqrt{GM_\oplus} R_\oplus^5. \quad (12)$$

In words: the rate of recession $\dot{a}$ falls very steeply with distance, as $a^{-11/2}$, because the tidal bulge weakens rapidly as the Moon moves away. The constant A gathers the tidal strength of the Earth, $k_2/Q_\oplus$, the mass ratio of the Moon to the Earth and the Earth’s size; a larger $k_2/Q_\oplus$ means faster recession.

Hence:

$$t(a_f) - t(a_0) = \frac{2}{13A} \left(a_f^{13/2} - a_0^{13/2} \right). \quad (13)$$

This is the time to travel from the release distance a_0 to a final distance a_f . Because of the high power 13/2, the term in a_f is overwhelmingly larger than the term in a_0 : nearly all the time is spent in the slow, late part of the journey, and the elapsed time hardly depends on where the Moon started. It is, however, inversely proportional to A , and therefore directly proportional to $Q_\oplus$.

For $k_2 = 0.299$, $Q_\oplus = 12$, $R_\oplus = 6,371$ km and $a_0 \simeq 18,200$ km, one obtains about 1.60 billion years to the present distance and about 3.70 cm/yr at that distance. The present rate is close to the observation, but the elapsed time does not reproduce 4.5 billion years. Increasing Q to match the elapsed time reduces the present rate by the same ratio. The time grows in proportion to Q and the present rate falls in proportion to $1/Q$, so no single constant value of Q can fit both.

This is a calibration test of a simple law, not a history of the ancient Earth. The law is also not valid by itself in the inclined ellipse of Part B or near synchronism. Varying dissipation histories and oceanic resonances are the subject of detailed orbital models [S8]: the Earth’s oceans respond to the tide according to the shape of their basins and the length of the day, and can pass through resonances in which the response is unusually strong, so the present rate need not be the long-term average. If the construction lasted a substantial part of the ancient period, the time available after detachment must be computed from that date, rather than automatically adding 4.5 billion years after detachment. In the series, the construction takes about 600 million years from the start of the bulge to release at 3.873 Ga, so the time available after release is about 3.9 billion years, not 4.5. Against 3.87 billion years the clock fits at $Q_\oplus \simeq 29$, where the present rate falls to about 1.5 cm/yr; the conclusion is the same.

Figure 3 shows the clock test for every constant Q between 10 and 60.

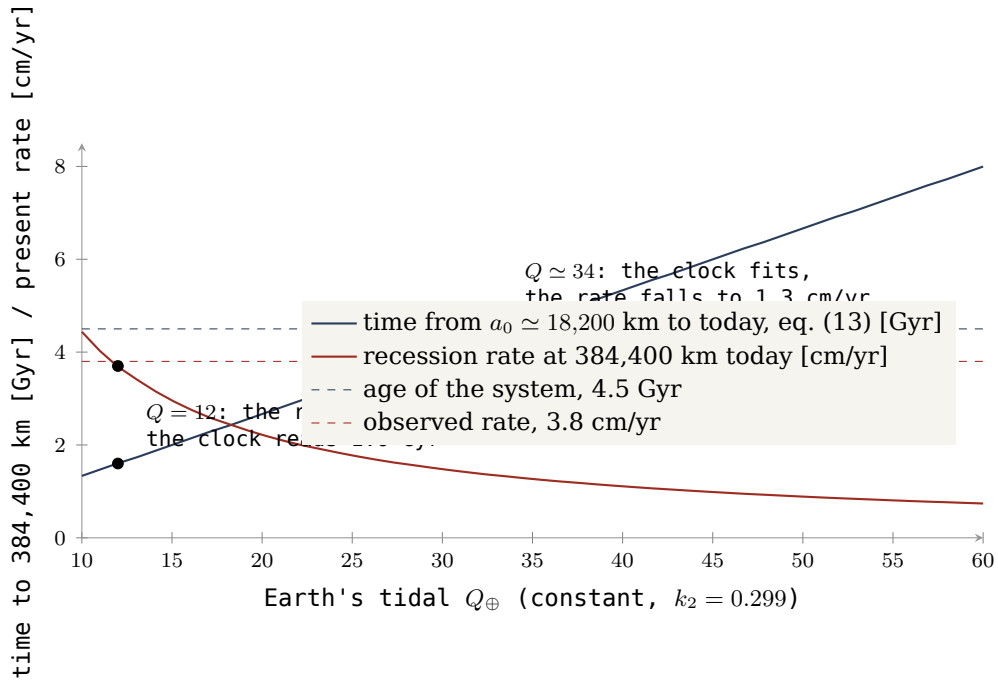


Figure 3. One constant- Q clock cannot fit both the present recession rate and the age of the system (Part D.3): the value that reproduces 3.8 cm/yr reaches the present distance in 1.6 Gyr, and the value that fills 4.5 Gyr gives a present rate three times too small. The dissipation history must have varied, which is what detailed models find [S8].

How to read Figure 3: the rising navy line is the travel time, which grows with Q ; the falling red line is the present recession rate, which drops with Q . Each solid line has to meet its dashed target, and the two meetings happen at different values of Q — near 12 for the rate and near 34 for the age. A history with a single constant Q cannot stand at both points at once.

What we now have: the tan-law carries the release latitude of 22.5° to 5.145° at the present distance, the observed value, and the clock test shows that the time this takes is set by a dissipation that varied through the history. The full answer is a coupled thermal-orbital integration from the release state, which is listed under Further work.

What has been proved and what will be tested?

THE PROBLEM AND THE SOLUTION

A mathematical identity, a possible orbital branch and an observational reconstruction must be distinguished. The table defines for each claim its conditions, quantitative result and test, so that the assessment of the paper does not rest on a visual or verbal match.

Parts A-D took the Moon from the moment of release to the question of the present orbit. This Part collects in one place what those Parts establish, and how each result can be checked. The distinction in the box above matters because the three kinds of result are checked in different ways. An identity, such as $i_0 = |\lambda|$, holds whenever its inputs hold; what is checked is only that it is applied in the right place. An orbital branch, such as the release state of Part B, is one solution among the solutions the equations allow, and what is checked is that the conditions it needs are met. A reconstruction, such as the present inclination, requires a full integration through the history of the system, and what is checked is that a permitted history reproduces the observations.

The table has one row for each claim. It is read from left to right: the claim; what it assumes; the number or sign it predicts; and how that prediction is checked. The tint of the rows follows the status of the claim, from top to bottom: navy for a geometric identity, cream for a result computed in this paper, red for a result that requires the integration listed under Further work. The rows therefore run from what is already fixed to what is still to be computed.

Claim	Assumptions	Quantitative expectation	Test
Source of the inclination — a geometric identity	Local release, eastward tangential velocity, latitude relative to the axis at release	$i_0 = \lambda $	Centre-of-mass vectors at the end of separation
The initial ellipse — two-body mechanics	Two bodies, $v_r = 0$, $0 < B < 2$	Equations (6)–(7)	Comparison with a short orbital integration

Claim	Assumptions	Quantitative expectation	Test
Initial recession — computed (§B.2)	The release conditions and tidal rheology defined in advance	$\langle \dot{a} \rangle > 0$	Power average from both tidal systems
Survival — Roche test passed (§B.2); deformation check open	Density, strength and thermal state consistent with the construction	The body survives the first perigees	Check of deformation, heating and mass loss
The present inclination — requires integration	A shared history of axes, Laplace, Cassini and dissipation	About 5.145° to the ecliptic, alongside the present eccentricity	Vector thermal-orbital integration
Distance and clock — requires integration	A detachment date and time-dependent dissipation consistent with each other	About 384,400 km and about 3.8 cm/yr today	Joint fit to the data and the time constraints
Geographic link — requires an independent reconstruction	Reconstruction of the ancient latitude of the centre of mass	An independent latitude, with an uncertainty range	Structural and rotational reconstruction not fitted to the orbit

In plain words, each claim and its test are as follows.

- *Source of the inclination.* The claim is that a body released while turning eastward with the Earth at latitude λ starts in an orbit tilted by λ to the equator of that time (§A.2). The test is the centre-of-mass velocity at the end of separation: the identity applies as long as that velocity carries no significant north-south component and no torque during separation turns the orbital plane. The identity is measured against the ancient equator, so it is not compared directly with today's 5.145° , which is measured against the ecliptic (Part C).
- *The initial ellipse.* The claim is that equations (6)–(7) give the size and shape of the first orbit from the release radius, the spin and the latitude (§B.1). Here $v_r = 0$ means that the body has no velocity directly away from or toward the Earth at release, and $0 < B < 2$ is the range of the speed ratio B for which the orbit is closed. The test is a short numerical integration of the same initial conditions, which should return the same perigee; a velocity added during separation, or the field of the flattened Earth, would require the ellipse to be recomputed.
- *Initial recession.* The claim is that, from the adopted release, the Moon moves outward from the first orbit: $\langle \dot{a} \rangle > 0$, where $\dot{a}$ is the rate of change of the orbit's size and the brackets mean its average over one orbit. The test averages the tidal power from both bodies, the tide raised on the Earth and the tide raised on the Moon. At the adopted release this average is positive (§B.2).
- *Survival.* The claim is that the Moon outlasts its first passages close to the Earth. The adopted release lies about 7,200 km beyond the rigid Roche limit of 10,314 km (§B.2), and the body is rigid and round at release (Paper 2, *LEF Mechanism in*

Principle, §B.7). The remaining check is that the combination of density, strength and temperature that keeps the body whole through deformation, heating and mass loss is one that the construction of Paper 2 and Paper 3 (*LEF Mechanism in Practice*) produces.

- *The present inclination.* The claim is that the 22.5° of release becomes the present 5.145° to the ecliptic. The reduced law of §D.1 already gives 5.145° at the present distance from the adopted release; the full test is a vector integration through the Laplace and Cassini transitions with a time-dependent dissipation (Further work, item 1). The integration is to reproduce the present inclination and the present eccentricity together.
- *Distance and clock.* The claim is that the Moon reaches about 384,400 km in the time available since release and recedes today at about 3.8 cm/yr. §D.3 shows that a single constant dissipation cannot do both; the test is a joint fit of distance, rate and time. The fit is to reach the present distance within the age allowed, with a dissipation history the Earth can have had.
- *Geographic link.* The claim is that the release latitude can be traced to a place on the Earth. It is tested by reconstructing the ancient latitude of the centre of mass from structure and rotation alone, without fitting it to the orbit. The identity $i_0 = |\lambda|$ of the first row (equation (3) of §A.2) does not depend on this reconstruction, because it does not depend on where the site was.

The modern coordinate 25° north, 165° east, proposed in Paper 3, is not a direct measurement of the ancient release latitude; this paper adopts $\lambda = 22.5^\circ$, the latitude of the centre of the contact area on the Moon. The three locations involved should be kept apart. The base of the pear stood near Kiritimati, $1.9^\circ\text{N } 157.5^\circ\text{W}$ (Paper 3), roughly on the equator of that time. The coordinate $25^\circ\text{N } 165^\circ\text{E}$ is read as the back of the pear. The release latitude 22.5° is the latitude of the centre of the contact area, about 3,000 km across, on which the Moon sat at the tip of the pear; it is used throughout this paper. 25° is a sensitivity case: the reduced law of §D.1 gives 5.788° from it, against 5.145° from 22.5° .

The width of the connection footprint is not a free range for choosing the latitude of the centre of mass, and the disappearance of a local gravitational perturbation is not proof that all dynamical information on the initial conditions was erased. In other words, the size of the contact area does not allow the latitude to be moved within it until the orbit fits; the latitude is set by where the centre of mass of the Moon lay. Reconstructing a specific source site is distinct from constraining families of initial orbits: the first asks where on the Earth the Moon left from; the second asks which release states lead to the present orbit, and it can be answered without the first.

Map of contribution and prior knowledge

The table separates what this paper takes from standard mechanics and the literature from what it adds. It is read row by row: the left column names a component of the argument, the right column says whether it is prior knowledge, an application of it, or the contribution of the model.

Component	Status
$\mathbf{h} = \mathbf{r} \times \mathbf{v}$, Keplerian elements and conservation of angular momentum	Standard mechanics; the derivation here is explicit
Inclination change by tides, resonances and encounters	Physics and mechanisms studied in the literature [S2–S8]
Source of inclination from release at non-zero latitude	Kinematic application; no claim of priority for a mathematical identity
Linking the latitude to the release branches and the budget of Paper 2, and to the construction limits of Paper 3	The model contribution of this work
Arrival from the construction to the present Moon	The result of the series; the full coupled integration is listed under Further work

The first three rows are standard: the orbital angular momentum per unit mass $\mathbf{h}$ (the cross product of position and velocity), the Keplerian elements that describe an orbit, and the known ways in which tides and encounters turn an orbital plane. What the model adds is the fourth row: it ties the latitude of release to the release states and angular-momentum budget computed in Paper 2 and to the limits of construction set in Paper 3, so that the starting orbit is fixed by the construction rather than chosen to fit. The last row is the aim of the series as a whole.

Conclusions

The paper reaches six conclusions. The first three follow from the release state and are fixed; the last three state what the reduced calculations show and what the full integration must do.

1. Local release at an eastward tangential velocity gives $i_0 = |\lambda|$ relative to the equator at the moment of release. This is a geometric source for an initial inclination in the LEF model. A body that leaves from a point off the equator, moving eastward with the Earth, moves in a plane that passes through the Earth's centre and through that point; this plane is tilted to the equator by the latitude of the point.
2. The same geometry gives the release radius: off the equator the corotation line, where the orbit would be a circle, lies at $r_c \cos^{-2/3} \lambda$; at a 5.77-hour day and $\lambda = 22.5^\circ$, 17,287 km. The Moon begins to separate as its centre of mass reaches this line and is released about 210 km beyond it, at $r_0 = 17,496$ km, when a little more than half of it has crossed; at that distance the reduced inclination and eccentricity tests of Paper 3 (§C.5a) meet. The corotation radius r_c (16,399 km) is where a body on the equator, turning with the Earth, already has the speed of a circular orbit. Off the equator the body is closer to the spin axis and so moves more slowly; it reaches circular speed a little farther out, by the factor $\cos^{-2/3} \lambda$, which is 1.0542 at 22.5° .
3. Released there, the Moon enters a nearly circular orbit ($e_0 \simeq 0.037$, the release point its perigee), is pushed outward by the tide from the first moment and lies about 7,200 km beyond the rigid Roche limit. The rigid Roche limit, 10,314 km, is the distance inside which the Earth's tide would pull apart a rigid body of the Moon's density.

4. The solution of the reduced decay law is a $\tan i$ law. It exposes sensitivity to the release conditions and does not cancel the dependence of the full system on rheology and cooling. From the release state it gives 5.145° at the present distance, the observed value (§D.1); the agreement is the result of a single reduced equation, and the full integration is still required.
5. A constant-dissipation tidal calculation that approaches the present recession rate does not by itself reproduce the elapsed time since release. A joint fit of distance, time, eccentricity and inclination is required. With $Q_\oplus = 12$ the present rate comes out at about 3.70 cm/yr, but the present distance is reached in about 1.60 billion years (§D.3); the dissipation of the Earth must have varied with time.
6. The paper defines the transition from construction to orbit as a quantitative programme; its inputs are all named, and the full thermal-orbital reconstruction is the first item of the Further work.

What we now have is a starting orbit fixed by the construction — $a_0 \simeq 18,200$ km, $e_0 \simeq 0.037$, $i_0 = 22.5^\circ$ — and a list of the calculations that lead from it to the present Moon.

Formulas for reproducing the tidal test

This appendix gives the formula behind the column “Initial terrestrial tidal contribution” of the table in §B.2, and a short code that reproduces the numbers of Parts B and D. The question the formula answers is simple: does the tide raised on the Earth push the Moon outward or draw it inward?

In the constant-time-lag framework, the contribution of the tide in the parent body to $\dot{a}$ is positive when [S4; application and calculation here]:

$$\frac{\omega \cos i}{n} > \frac{f_1(e)}{(1 - e^2)^{3/2} f_2(e)}, \quad n = \sqrt{\frac{K}{a^3}}. \quad (14)$$

The “constant time lag” means that the Earth’s tidal bulge responds to the Moon’s pull with a fixed small delay; during that delay the spinning Earth carries the bulge forward, ahead of the Moon or behind it. The symbols are: ω the Earth’s spin rate; i the inclination of the orbit to the equator, so that $\omega \cos i$ is the part of the spin that turns in the plane of the orbit; n the mean motion, the Moon’s average angular speed around its orbit, which by Kepler’s third law follows from the semi-major axis a and $K = G(M_\oplus + m)$; and e the eccentricity. The left side therefore compares how fast the Earth turns, in the plane of the orbit, with how fast the Moon goes round. In words, the condition says: the Earth’s tide pushes the Moon outward when the Earth turns faster than the Moon orbits, by a margin that depends on the eccentricity.

For a circular orbit, $e = 0$, the right side equals 1 and the condition reduces to $\omega \cos i > n$: the bulge runs ahead of the Moon and pulls it forward, as for a satellite beyond the corotation line. In an eccentric orbit the Moon moves faster than its average near perigee, where it is also closest and the tide is strongest; the right side then exceeds 1, and the Earth must turn correspondingly faster for the net effect to be outward. The two eccentricity functions measure this:

$$f_1(e) = 1 + \frac{31}{2}e^2 + \frac{255}{8}e^4 + \frac{185}{16}e^6 + \frac{25}{64}e^8, \quad f_2(e) = 1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{5}{16}e^6. \quad (15)$$

Both equal 1 for a circle and grow with e . Applied to the rows of §B.2, the test gives $0.7526 < 1.1962$ for release on the equatorial line (the orbit contracts), equality on the corotation line $x = 1.0542$ (the tide is neutral), $1.0769 > 1.0128$ at the adopted release $x = 1.0669$, about 210 km beyond the line (the orbit expands from the first moment; an earlier version, with release about 100 km beyond the line at $x = 1.0603$, gave $1.0357 > 1.0029$), and $1.1660 > 1.0532$ at $x = 1.08$ (the orbit expands). On the corotation

line the equality is exact by construction: the circular radius of equation (7a) is the radius at which $\omega \cos \lambda = n$.

The polynomials are not universal for every dissipation law. To separate mathematical verification from a material assumption, the sign of the power was also checked by integrating the tangential and radial force along one orbit in the same model. The calculation does not include tides in the Moon, a solar torque or an active connection. The sign of the effect is thus confirmed independently of the series expansion; what remains an assumption is the constant-time-lag model itself.

Below is short code for reproducing the ellipses, the tidal sign and the inclination illustration from the coupled initial conditions of Paper 2. It is not presented as a 4.5-billion-year integrator.

```
from math import pi, sin, cos, tan, atan, sqrt, radians, degrees
G, M, m = 6.674e-11, 5.972e24, 7.35e22
K = G * (M + m)
lat = radians(22.5)
# r0 [km], Earth day at release [h]: coupled outputs of Paper 2
cases = [(16398.5, 5.77), # on the equatorial line: release at apogee
         (17287.3, 5.77), # corotation line: circular, tide neutral
         (17496.3, 5.77), # adopted: about 210 km beyond the line
         (17710.4, 5.77)] # 8% beyond the line
for r_km, day_h in cases:
    r = 1000 * r_km
    omega = 2*pi/(3600*day_h)
    B = omega**2*r**3*cos(lat)**2/K
    a, e = r/(2-B), abs(1-B)
    n = sqrt(K/a**3)
    f1 = 1 + 15.5*e**2 + 31.875*e**4 + 11.5625*e**6 + .390625*e**8
    f2 = 1 + 7.5*e**2 + 5.625*e**4 + .3125*e**6
    lhs = omega*cos(lat)/n
    rhs = f1/((1-e**2)**1.5*f2)
    toy_i = degrees(atan(tan(lat)*sqrt(a/384400e3)))
    print(a/1000, e, a*(1-e)/1000, lhs, rhs, toy_i)
```

The code follows the paper step by step. It takes four release radii at the 5.77-hour day of release and latitude 22.5°: on the equatorial corotation line (16,398.5 km), on the corotation line for 22.5° (17,287.3 km; an earlier version gave 17,286.9 km, which leaves a residual $e \approx 8 \times 10^{-5}$), at the adopted release about 210 km beyond it (17,496.3 km; an earlier version used 17,387.3 km, about 100 km beyond it), and 8% beyond the equatorial line (17,710.4 km). For each it computes the speed ratio B of equation (6), the squared ratio of the speed carried by the rotation to the speed of a circular orbit at the same distance; then the semi-major axis a and eccentricity e of equation (7); the mean motion n ; the eccentricity functions of equation (15), written as decimals (15.5 is 31/2, 31.875 is 255/8, 11.5625 is 185/16, 0.390625 is 25/64, 7.5 is 15/2, 5.625 is 45/8, 0.3125 is 5/16); and the two sides of equation (14). The last line applies the reduced law of equation (11) with $\epsilon = 0$ from a to the present distance of 384,400 km. Each printed line gives a in km, e , the perigee in km, the two sides of the tidal test, and the final angle. The output reproduces the rows of §B.2 (14,304 km, 0.1464, 12,209 km; 17,287 km, 0, 17,287 km; 18,163 km, 0.0367, 17,496 km; 19,151 km, 0.0752, 17,710 km), the tidal test 1.0769 > 1.0128 at the adopted release, and the angles 5.145° and 5.282° of §D.1.

Scope of verification and availability

This appendix states what was checked, how, and what the checks do not cover, so that a reader can judge how far each number of the paper has been verified.

The original file was read in full. The algebra of the decay equation was also checked against the original page. The structure and spin branches in the table were taken from the calculations of Paper 2, and their conversion to orbital elements was rechecked. In the check of Paper 2, an independent Newtonian integration of the reference orbit reproduced the perigee and the return to the starting point; the relative return error in position was about 1.6×10^{-10} . That is, after one full orbit the integrated body came back to its starting point to within less than one part in a billion of its distance, which confirms that the orbit computed from equations (6)–(7) is the orbit that Newtonian gravity produces. This verification tests the monopole kinematics, not the reality of the connection or the validity of the rheology. “Monopole” means that each body is treated as a point mass; the check confirms the orbital arithmetic, while the physical assumptions remain those named in Part E.

The names of the sources and their links were checked, and the claims attributed to them were limited to content accessible in the source pages and their abstracts; no equation-by-equation review of every cited paper is claimed. The document includes the code for the follow-on calculations. The complete structural solutions belong to the supplementary material of Paper 2; a full coupled thermal-orbital integration is not included and is not reported as if performed.

Further work — openings for research

The calculations of this paper fix the starting point; the items below are the calculations that carry it forward. Each names an input this paper supplies and the result the calculation would return.

1. **A coupled thermal-orbital integration from the release state.** The initial conditions of the release state of Part B — release at $r_0 = 17,496$ km as perigee, $a_0 \simeq 18,200$ km, $e_0 \simeq 0.037$, $i_0 = 22.5^\circ$, the 5.77-hour day and J_2 scaled to it as $J_{2,\text{now}}(\omega/\omega_{\text{now}})^2$ — are fixed; what is needed is an integration through the Laplace-plane transition, the Cassini state and a time-dependent dissipation in both bodies, of the kind that existing secular integrators perform. Its result is the prediction of the present inclination. Here J_2 measures the Earth’s equatorial bulge; for a fluid body the bulge grows with the square of the spin rate, hence the scaling. The Laplace-plane transition is the distance at which the plane about which the orbit precesses changes from the Earth’s equator, which governs close in, to the ecliptic, which governs far out (§C.1). The Cassini state is the steady arrangement in which the Moon’s spin axis, the normal to its orbit and the normal to the ecliptic turn together (§C.3). A secular integrator follows the slow, orbit-averaged evolution of the orbit over billions of years.
2. **The dissipation history of the Earth.** No constant Q fits both the present rate and the age (§D.3). Ocean-resonance models of the early Earth, run from the 5.77-hour day of release (rather than the 6.4-hour day of the earlier, pear-alone state), would provide the rate function required by the integration. Q measures how little of the tidal energy the Earth dissipates in each cycle: a low Q means strong dissipation and fast recession. Ocean resonance occurs when the natural sloshing period of an ocean basin approaches the tidal period; the dissipation then rises sharply, and it changes as the day lengthens and the continents move [S8].
3. **The Earth-Moon axes.** The angle of about 3.8° between the Earth’s spin and the total angular

momentum at release (§A.3) belongs to the Earth-Moon axes system, future work outside this series, about which no claim is made here.

4. **The magma-ocean switch.** The survival of the inclination depends on whether the lunar magma ocean had solidified before 20 Earth radii; the constraints on this from the models of [S6] and [S7] are discussed in §D.2. A thermal model of a Moon with a one-sided lid — thick on the far side, thin on the near side, whose maria were still hot at detachment (Paper 3) — would give the solidification time for this body rather than a symmetric one. The magma ocean is the molten outer layer of the young Moon; while it is liquid, the Moon dissipates tidal energy strongly and the inclination is damped faster. A lid of uneven thickness cools at a different rate from the uniform shell assumed in the published models, so the time at which the switch occurs must be computed for this body.
5. **The fossil figure.** A body that went through outward rounding, flattening and inward rounding while it sat, as a ball squashed on the tip of the pear, carries its figure from that stage, not from a later tidal lock. The relation between the geometry of the squashed ball on the tip of the pear (Paper 3) and the present degree-2 shape is a calculation not yet made. The fossil figure is the slight departure of the Moon’s shape from a sphere that its rigid outer shell has kept from an earlier time; the degree-2 shape is the largest-scale part of that departure, its overall elongation and flattening, as measured in the gravity field. In the LEF model the flattening and rounding of the lunar body took place while it was still attached (Paper 7, *The Poles*); the calculation would test whether the shape left by that stage matches the present one.

Code availability

All numerical results of this paper are reproduced by the scripts of the LEF reproduction package (Python 3, numpy, scipy), which prints the values of each section from the inputs stated in the text. The package is attached to this preprint as supplementary material (LEF_code_and_data.zip). An earlier version of the series and of the package is deposited at doi:10.5281/zenodo.22743615.

References

- [1] Duran, Z. (2026). *The Model*. LEF, Paper 1.
- [2] Duran, Z. (2026). *LEF Mechanism in Principle — Corotation and Angular Momentum*. LEF, Paper 2. Integrated working manuscript, including the release-latitude and orbital-survival calculations.
- [3] Duran, Z. (2026). *LEF Mechanism in Practice — Pressure and the Pear*. LEF, Paper 3. Integrated working manuscript; the connection and the release margin beyond corotation.
- [S1] Williams, D. R. (2024). *Moon Fact Sheet*. NASA Goddard Space Flight Center, NSSDCA. <https://nssdc.gsfc.nasa.gov>
- [S2] Pahlevan, K., & Morbidelli, A. (2015). *Collisionless encounters and the origin of the lunar inclination*. *Nature*, 527, 492–494. Authors’ copy: arXiv:1603.06515.
- [S3] Čuk, M., Lock, S. J., Stewart, S. T., & Hamilton, D. P. (2021). *Tidal Evolution of the Earth-Moon System with a High Initial Obliquity*. Authors’ copy: arXiv:2107.03353.
- [S4] Correia, A. C. M., Laskar, J., Farago, F., & Boué, G. (2011). *Tidal evolution of hierarchical and inclined systems*. *Celestial Mechanics and Dynamical Astronomy*. DOI: 10.1007/s10569-011-9368-9. Authors’ copy: arXiv:1107.0736.
- [S5] Tremaine, S., Touma, J., & Namouni, F. (2009). *Satellite dynamics on the Laplace surface*. *The Astronomical Journal*, 137, 3706–3717. Authors’ copy: arXiv:0809.0237.
- [S6] Downey, B. G., Nimmo, F., & Matsuyama, I. (2023). *The thermal-orbital evolution of the Earth-Moon system with a subsurface magma ocean and fossil figure*. *Icarus*, 389, 115257. Authors’ copy: arXiv:2209.00935.
- [S7] Chen, E. M. A., & Nimmo, F. (2016). *Tidal dissipation in the lunar magma ocean and its effect on the early evolution of the Earth-Moon system*. *Icarus*, 275, 132–142.
- [S8] Farhat, M., Auclair-Desrotour, P., Boué, G., & Laskar, J. (2022). *The resonant tidal evolution of the Earth-Moon distance*. *Astronomy & Astrophysics*, 665, L1.

Use of AI tools: The author used AI assistants (Claude, Anthropic; ChatGPT, OpenAI; Google's AI search assistant) to write and run the computation scripts, to analyse the catalogues and maps, to prepare figures and tables, to discuss the scenarios, and for language editing and translation. The model and its conclusions are the author's. The author checked the content and takes full responsibility for it.